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THE EXISTENCE OF A NON-NEGATIVE SOLUTION OF AN ORDINARY DIFFERENTIAL EQUATION ARISING IN ELECTROMAGNETIC THEORY

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ABSTRACT

The existence of a non-negative solution of the boundary-value problem

$$y'' + \frac{2}{r}y' + [y - (1 + \frac{2}{r^2})]y = 0 \quad ,$$

$$y(0) = 0, \quad y(\infty) = 0 \; ,$$

is proved by a shooting argument. The equation arises in an electromagnetic theory for strong interaction in charged media.

AMS(MOS) Subject Classification - 34B15

Key Words - Ordinary differential equations, boundary-value problems, shooting arguments.

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THE EXISTENCE OF A NON-NEGATIVE SOLUTION OF AN ORDINARY DIFFERENTIAL EQUATION ARISING IN ELECTROMAGNETIC THEORY

J. B. Mc Leod

1. Introduction

We are interested in the boundary-value problem associated with

$$y'' + \frac{2}{r} y' + [y - (1 + \frac{2}{r^2})]y = 0 \tag{1.1}$$

where the function $y(r)$ satisfies the boundary conditions

$$y(0) = 0, \quad y(\infty) = 0 . \tag{1.2}$$

This problem arises in an electromagnetic theory of strong interaction which is discussed by Bergström in [1], although the equation (1.1) is not specifically derived in [1], and I am indebted to Seymour Parter for bringing the problem to my attention. We want to show the existence of a non-negative solution to this problem, and the method of attack is to "shoot" from the origin. To this end we prove the following sequence of lemmas, which we state now with the proofs given later.

Lemma 1. *Given any* $\alpha > 0$, *there exists a unique solution of* (1.1) *which has the asymptotic behaviour*

$$y(r) \sim \alpha r \quad \text{as} \quad r \to 0 ,$$

and plainly this solution is non-negative initially (*i.e. for* r *sufficiently small*).

Lemma 2. *Any solution* $y(r)$ *of* (1.1) *which is initially non-negative must possess one of three mutually exclusive properties*:

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(i) it remains non-negative for only a finite range of values of r ;

(ii) it remains non-negative for all r, and $y(r) \to 1$ as $r \to \infty$;

(iii) it remains non-negative for all r, and $y(r) \to 0$ as $r \to \infty$.

Lemma 3. The set of $\alpha(>0)$ for which the solution given by Lemma 1 has the property (i) of Lemma 2 is denoted by S_1 and is open in the topology of the positive semi-axis.

Lemma 4. The set of $\alpha\ (>0)$ for which the solution given by Lemma 1 has the property (ii) of Lemma 2 is denoted by S_2 and is open in the topology of the positive semi-axis.

Lemma 5. If $\alpha\,(>0)$ is sufficiently large, then the solution given by Lemma 1 has the property (i) of Lemma 2, so that $\alpha \in S_1$.

Lemma 6. If $\alpha(>0)$ is sufficiently small, then the solution given by Lemma 1 has the property (ii) of Lemma 2, so that $\alpha \in S_2$.

Once Lemmas 1-6 are proved, as they are below, then the following *existence theorem becomes an easy consequence.*

Theorem. There exists a non-negative solution to the boundary-value problem (1.1)-(1.2), the solution being in fact strictly positive except at $r = 0$.

Proof. Lemmas 3-6 imply that the sets S_1, S_2 are open and non-empty. *Since the positive semi-axis is connected,* not all $\alpha > 0$ can belong to S_1, S_2 . Hence there exists at least one value of α which is in neither S_1 nor S_2, and for which therefore the corresponding solution given by

Lemma 1 has the property (iii) of Lemma 2. This completes the proof, except for the trivial observation that the solution must be strictly positive for $r > 0$, since if it were not it would have to touch the value 0, at $r = r_0$, say, when $y(r_0) = y'(r_0) = 0$, and this implies from (1.1) that $y \equiv 0$, contradicting $\alpha > 0$.

It is perhaps worth remarking that there is a difference in character between Lemmas 1-4 on the one hand and Lemmas 5-6 on the other. Thus Lemma 1 is a statement about the asymptotic behaviour of solutions of the equation at the origin, and this is something that an analyst can expect to discuss relatively easily, even for a much more complicated equation than (1.1), provided only that the coefficients in the equation are, asymptotically at the origin, of a fairly simple character. In the same way, Lemma 2 is concerned with the asymptotic behaviour at infinity, and Lemmas 3-4 are statements about the continuity of the solution in the initial parameter α. All these lemmas are therefore accessible to analytical proof, again even for a more complicated equation; and conversely, since Lemma 2 is a statement about all (and so an infinite number of) solutions of a certain kind, and since infinity and continuity are not concepts which the computer recognizes, it is difficult to see how these lemmas could be established other than analytically.

The purpose of Lemmas 5-6, however, is only to show that there exists one solution in the set S_1 and another solution in the set S_2. (More than this is stated (and proved) in these lemmas, but the existence

of one solution in each set is all that is required in the application.) To establish such results, a solution has to be followed, not only for small and large r, but also in general for moderate r as well, and in the case of a complicated equation, this may well prove impossible for the analyst. Even in the present case the analysis is delicate. On the other hand, since we are concerned with only a finite number of solutions, it may well be possible for the computer to follow these solutions and, with the aid of interval analysis or some comparable argument, prove that the solutions so followed satisfy the inequalities that place them in the required sets.

The point of these remarks is that a shooting argument thus breaks naturally into two parts, one of which should be accessible to the analyst and the other to the computer. It becomes therefore a very powerful tool for the proof of existence of solutions of two-point boundary-value problems.

2. Proof of Lemma 1

We can write (1.1) in the form

$$y'' + \frac{2}{r} y' - \frac{2}{r^2} y = (1 - y)y \quad , \tag{2.1}$$

and since two linearly independent solutions of

$$y'' + \frac{2}{r} y' - \frac{2}{r^2} y = 0$$

are $y = r$, $y = 1/r^2$, the variation of constants formula shows that (2.1) is equivalent to the integral equation

$$y = Ar + B/r^2 + \frac{1}{3} \int_{r_0}^{r} \left(r - \frac{t^3}{r^2}\right) \{1 - y(t)\} y(t) dt$$

where, for any fixed $r_0 > 0$, all solutions of (2.1) which exist in some neighbourhood of $r = r_0$ are given by arbitrary choices of the constants A, B. In fact, since we are interested in solutions for which $y \sim \alpha r$, we can actually choose $r_0 = 0$, since the integral continues to have meaning then, and we shall then have to choose $B = 0$, $A = \alpha$. Thus such a solution satisfies the equation

$$y = \alpha r + \frac{1}{3} \int_0^r \left(r - \frac{t^3}{r^2}\right) \{1 - y(t)\} \, y(t)dt \quad , \tag{2.2}$$

and the usual approach of solution by iteration shows that this equation has one and only one solution for which $y \sim \alpha r$, and so the lemma is proved.

3. Proof of Lemma 2

We first note from (2.2) that

$$y \leq \alpha r + \frac{1}{3} \int_0^r \left(r - \frac{t^3}{r^3}\right) y(t)dt \quad ,$$

from which it follows that $y \leq Y$, where Y is the solution of the integral equation

$$Y = \alpha r + \frac{1}{3} \int_0^r \left(r - \frac{t^3}{r^2}\right) Y(t)dt \quad , \tag{3.1}$$

with $Y \sim \alpha r$ as $r \to 0$. But it is easy to deduce from (3.1) that Y satisfies a linear differential equation, and so exists for all r and is bounded in any compact set. Since a non-negative solution y of (2.2) is bounded below by 0 and above by Y, it follows that y exists for all r and is bounded in any compact set. Hence, if alternative (i) in the statement of the lemma does not hold, then certainly y exists for all r and we have only to prove that $y \to 1$ or $y \to 0$ as $r \to \infty$.

We therefore assume for the remainder of the proof that alternative (i) does not hold, and show first that, if y is ultimately monotonic, then $y \to 1$ or $y \to 0$. For if we suppose for contradiction that $y(\infty) = k \neq 0, 1$, where $y(\infty)$ exists since y is monotonic, then either $0 < k < 1$ or $k > 1$ (with possibly $k = +\infty$). If $0 < k < 1$, (2.1) gives

$$(r^2 y')' = -r^2 y\{y - (1 + 2/r^2)\} ,$$

so that

$$(r^2 y')' \sim r^2 k(1-k) ,$$

$$r^2 y' \sim \frac{1}{3} r^3 k(1-k) ,$$

$$y' \sim \frac{1}{3} rk(1-k) ,$$

which contradicts $y \to k$, and a similar argument disposes of the case $k > 1$.

Thus if y is ultimately monotonic, then $y \to 1$ or $y \to 0$. It remains to discuss the case where y is not ultimately monotonic. Then multiplying (2.1) by y' and integrating from some value $r_0 > 0$, we have

$$[\tfrac{1}{2} y'^2]_{r_0}^{r} + \int_{r_0}^{r} \frac{y'^2(t)}{t} dt + [\tfrac{1}{3} y^3]_{r_0}^{r} - [\tfrac{1}{2} y^2]_{r_0}^{r} - 2 \int_{r_0}^{r} \frac{y(t)y'(t)}{t^2} dt = 0 . \qquad (3.2)$$

Now

$$y(r) = \int_0^r y'(t)dt$$

$$\leq r^{\frac{1}{2}} \{\int_0^r y'^2(t)dt\}^{\frac{1}{2}}, \quad \text{by Cauchy-Schwarz.}$$

Hence

$$2\int_{r_0}^{r} \frac{y(t)\,y'(t)}{t^2}\,dt = \left[\frac{y^2}{t^2}\right]_{r_0}^{r} + 2\int_{r_0}^{r} \frac{y^2(t)}{t^3}\,dt$$

$$\leq \left[\frac{y^2}{t^2}\right]_{r_0}^{r} + 2\int_{r_0}^{r} \left(\int_0^t y'^2(u)\,du\right)\frac{dt}{t^2}$$

$$= \left[\frac{y^2}{t^2}\right]_{r_0}^{r} - 2\left[\frac{1}{t}\int_0^t y'^2(u)\,du\right]_{r_0}^{r} + 2\int_{r_0}^{r} \frac{y'^2(t)}{t}\,dt ,$$

so that (3.2) gives

$$\left[\frac{1}{2}y'^2 + \frac{1}{3}y^3 - \frac{1}{2}y^2 - \frac{y^2}{t^2} + \frac{2}{t}\int_0^t y'^2(u)\,du\right]_{r_0}^{r} \leq 0 . \qquad (3.3)$$

Since y^3 dominates y^2 for large y, this last inequality plainly implies bounds on both y and y' as $r \to \infty$.

Now we are assuming that y oscillates, and we may further suppose that it oscillates finitely, i.e. that $y(\infty)$ does not exist. For if $y(\infty)$ exists, then the argument used when y is monotonic will show that either $y(\infty) = 0$ or $y(\infty) = 1$. From the now-established boundedness of y and y', we see from (3.2) that

$$\int_{r_0}^{r} \frac{y'^2(t)}{t}\,dt \qquad (3.4)$$

is bounded as $r \to \infty$, and since it is monotonic, it must converge. We thus have

$$\frac{1}{2} y'^2 + \frac{1}{3} y^3 - \frac{1}{2} y^2 \to L, \text{ say, as } r \to \infty , \tag{3.5}$$

and three cases now arise, depending on whether $L = 0$; $L = -1/6$; $L \neq 0$, $-1/6$. (The cases $L = 0$, $L = -1/6$ are the cases in which the roots of

$$\frac{1}{3} y^3 - \frac{1}{2} y^2 - L = 0$$

are not distinct.)

If $L = -1/6$, then the possible extreme values of y (in the limit as $r \to \infty$), i.e. the roots of the equation

$$\frac{1}{3} y^3 - \frac{1}{2} y^2 + \frac{1}{6} = 0 ,$$

are readily verified to be 1 (twice) and $-1/2$. Since y is non-negative, this implies that $y \to 1$. (Alternatively, if $L = -1/6$, then (3.5) gives

$$\frac{1}{2} y'^2 + \frac{1}{6}(2y + 1)(y - 1)^2 \to 0 ,$$

from which it immediately follows that $y' \to 0$, $y \to 1$.)

If $L \neq 0$, $-1/6$, then the roots of

$$\frac{1}{3} y^3 - \frac{1}{2} y^2 - L = 0$$

are distinct, and none of them are 0 or 1. In the limit as $r \to \infty$, y oscillates between two of these roots as extreme values, and in the course of a complete oscillation, y' is small if and only if y is near these extreme values. Further, from (1.1), y'' is not small when y is near the extreme values, since the extreme values are neither 0 nor 1. Hence

in the course of a complete oscillation, $y'(r)$ is small if and only if r is near a value for which $y'(r) = 0$, and so $y'(r)$ is not small for most of the r-range of the oscillation. From this it follows easily that

$$\int_{r_0}^{r} \frac{y'^2(t)}{t}\, dt$$

is not bounded as $r \to \infty$, and this is a contradiction to what we proved at (3.4).

If $L = 0$, the possible extreme values are 0 (twice) and $3/2$. There is no immediate reason (as there was with $L = -1/6$) why the solution should not oscillate (in the limits as $r \to \infty$) between 0 and $3/2$, and the argument employed when $L \neq 0, -1/6$ does not apply because it is *prima facie* possible for the solution to remain small (y, y', y'' all small) for long intervals of r, increasingly long every time the oscillation is repeated, and this situation does not necessarily contradict the boundedness of (3.4).

To argue that the solution cannot oscillate indefinitely between 0 and $3/2$, we must therefore look more deeply. Consider the function

$$F(r) = \frac{1}{2}y'^2 + \frac{1}{3}y^3 - \frac{1}{2}y^2 \quad . \tag{3.6}$$

We are supposing (for contradiction) that $F(r) \to 0$ as $r \to \infty$, while y oscillates indefinitely between 0 and $3/2$. As r increases from a local maximum r_0 of y to a local minimum r_1, so that $y' < 0$ in (r_0, r_1), it is clear from (3.2) that

$$F(r_1) - F(r_0) < 0 \quad ,$$

so that F decreases between the maximum and the minimum. Further, at a local minimum of y (at least once r is sufficiently large that y is small at the local minimum), $F < 0$, since $y' = 0$ and $-\frac{1}{2}y^2$ dominates $\frac{1}{3}y^3$. If therefore we can show also that F decreases as r increases from a local minimum of y to a local maximum, then we are done, since F (evaluated at local maxima and minima of y) will be negative decreasing, and so cannot tend to 0.

The decrease of F between a local minimum and a local maximum of y will be established from (3.2) if we can show that

$$\int_{r_0}^{r_1} \frac{y'^2(t)}{t} - \int_{r_0}^{r_1} \frac{y(t)\,y'(t)}{t^2}\,dt > 0 , \tag{3.7}$$

r_0 being the minimum and r_1 the maximum. We start by demonstrating that, given $\sigma > 0$, small but fixed, we have

$$y(r) < Ky(s), \qquad r_0 \leq s < r \leq r_1 , \; r - s \leq \sigma , \tag{3.8}$$

where K is a (positive-valued) function of σ but not of r_0, r_1, i.e. K is independent of how far out the oscillation is. For from (1.1) we have, for $t \in [r_0, r_1]$,

$$y''(t) < 2y(t) ,$$

at least if $r_0 \geq \sqrt{2}$, which we may suppose, and then by integration

$$[\tfrac{1}{2} y'^2]_{r_0}^{t} < [y^2]_{r_0}^{t}$$

and

$$y'(t) < y(t)\sqrt{2} . \qquad (3.9)$$

Then $$y(r) - y(s) = (r-s)\, y'(\xi), \qquad s < \xi < r ,$$

$$< \sigma\sqrt{2}\; y(r) ,$$

from which (3.8) follows.

We next establish that

$$y'(t) > K y(t) \quad \text{for} \quad t \in [r_0 + \sigma,\ r_1 - \sigma] , \qquad (3.10)$$

K again a (positive-valued) function of σ but not of r_0, r_1. (K may not be the same function as before, but it will cause no confusion to use the same notation for both.) For, since $F(r) \to 0$ as $r \to \infty$, we can readily deduce that y' is bounded away from 0 (and so a fortiori $y' > Ky$) so long as y is bounded away from $0, 3/2$. Since y'' is bounded away from 0 when y is near $3/2$, the statement that y is bounded away from $3/2$ is equivalent to the statement that r is bounded away from r_1, and so we certainly have

$$y'(t) > K\, y(t)$$

if $t \in [r_0 + \sigma, r_1 - \sigma]$ and $y(t)$ is bounded away from 0. But if $\tau \in [r_0, r_1]$ and $y(\tau)$ is near 0, say $y(\tau) < \frac{1}{2}$, then, from (1.1),

$$y''(\tau) > \frac{1}{2} y(\tau) - \frac{2y'(\tau)}{\tau} \geq \frac{1}{2}\, y(\tau) - \frac{2y'(\tau)}{r_0} ,$$

so that, integrating over $[r_0, r]$ with $r \geq r_0 + \sigma$, we have

$$y'(r) > \frac{1}{2}\int_{r_0}^{r} y(t)dt - \frac{2}{r_0}\{y(r) - y(r_0)\}$$

$$> \frac{1}{2}\int_{r-\sigma}^{r} y(t)dt - \frac{2}{r_0}\, y(r)$$

$$> \frac{1}{2}\,\sigma\, y(r-\sigma) - \frac{2}{r_0}\, y(r)$$

$$> \left(\frac{\sigma}{2K} - \frac{2}{r_0}\right) y(r), \qquad \text{from (3.8)},$$

which proves the required result since σ (and K) are independent of r_0 and r_0 can be chosen so large that

$$\frac{2}{r_0} < \frac{\sigma}{2K} .$$

If we now consider the difference in (3.7), we see that it exceeds

$$\int_{r_0+\sigma}^{r_1-\sigma} \left(\frac{y'^2}{t} - \frac{yy'}{t^2}\right)dt - \int_{r_0}^{r_0+\sigma} \frac{yy'}{t^2}\,dt - \int_{r_1-\sigma}^{r_1} \frac{yy'}{t^2}\,dt$$

$$> \{K(r_0+\sigma) - 1\}\int_{r_0+\sigma}^{r_1-\sigma} \frac{yy'}{t^2}\,dt - \int_{r_0}^{r_0+\sigma} \frac{yy'}{t^2}\,dt - \int_{r_1-\sigma}^{r_1} \frac{yy'}{t^2}\,dt ,$$

from (3.10),

$$> \{K(r_0+\sigma) - 1\}\left\{\int_{r_0+\sigma}^{r_0+2\sigma} + \int_{r_1-2\sigma}^{r_1-\sigma}\right\} \frac{yy'}{t^2}\,dt -$$

$$- \int_{r_0}^{r_0+\sigma} \frac{yy'}{t^2}\,dt - \int_{r_1-\sigma}^{r_1} \frac{yy'}{t^2}\,dt . \qquad (3.11)$$

Considering sufficiently the integrals involving r_0, we see that

$$\int_{r_0+\sigma}^{r_0+2\sigma} \frac{yy'}{t^2}\,dt > K\int_{r_0+\sigma}^{r_0+2\sigma} \frac{y^2}{t^2}\,dt, \qquad \text{from (3.10)},$$

$$> K\sigma \frac{y^2(r_0+\sigma)}{(r_0+2\sigma)^2},$$

while

$$\int_{r_0}^{r_0+\sigma} \frac{yy'}{t^2}\,dt < \sqrt{2}\int_{r_0}^{r_0+\sigma} \frac{y^2}{t^2}\,dt, \qquad \text{from (3.9)},$$

$$< \sigma\sqrt{2}\,\frac{y^2(r_0+\sigma)}{r_0^2},$$

from which it is clear (for r_0 sufficiently large) that the contribution from these integrals to (3.11) is positive. With similar arguments for the integrals involving r_1, we see that (3.17), and so the lemma, are proved.

4. Proof of Lemma 3

This is an immediate consequence of the fact that the solution $y(r, \alpha)$ of (2.2) is continuous in α. Thus if $y(r_0, \alpha_0) < 0$, we must also have $y(r_0, \alpha) < 0$ for α sufficiently close to α_0.

5. Proof of Lemma 4

Let us suppose that for $\alpha = \alpha_0$ the corresponding solution $y(r, \alpha_0)$ given by Lemma 1 has the property that $y(r, \alpha_0) \to 1$ as $r \to \infty$. We have to

show that, for α sufficiently close to α_0, we still have $y(r,\alpha) \to 1$ as $r \to \infty$.

Since $y(r,\alpha_0) \to 1$, we know that the quantity $F(r,\alpha_0)$ introduced in (3.6) (we now make explicit its dependence on α) has the property that

$$F(r,\alpha_0) \to -\frac{1}{6} \quad \text{as} \quad r \to \infty ,$$

and that (from the investigation of the case $L = -1/6$ in Lemma 2) $y(r,\alpha_0) \to 1$, $y'(r,\alpha_0) \to 0$. It is then an easy consequence that the expression in [. . .] in (3.3), which we will denote by $G(t,\alpha)$, has the property that

$$G(t,\alpha_0) \to -\frac{1}{6} \quad \text{as} \quad t \to \infty .$$

Hence, given $\varepsilon > 0$, we can find r_0 *sufficiently* large that

$$G(r_0,\alpha_0) \le -\frac{1}{6} + \varepsilon ,$$

and so, using continuous dependence on α, for α sufficiently close to α_0, we have

$$G(r_0,\alpha) \le -\frac{1}{6} + 2\varepsilon .$$

From (3.3) we can deduce for $r \ge r_0$ that

$$G(r,\alpha) \le -\frac{1}{6} + 2\varepsilon , \tag{5.1}$$

and it is then clear that $y(r,\alpha)$ cannot become zero for a finite value of r or have $y(\infty,\alpha) = 0$; for if either of these alternatives were to occur, then $G(r,\alpha)$ would become non-negative either at a finite value of r or in the

limit as $r \to \infty$, and this contradicts (5.1) with $\varepsilon < 1/12$. The only remaining possibility is that $y(r,\alpha) \to 1$ as $r \to \infty$, and the lemma is proved.

6. Proof of Lemma 5

We start by making a change of variables in (1.1) to take account of the fact that α is large. Thus we set

$$t = \alpha^{1/3} r, \quad Y(t) = \alpha^{-2/3} y(r) ,$$

and (1.1) becomes

$$Y'' + \frac{2}{t} Y' + \{Y - (\alpha^{-2/3} + \frac{2}{t})\} Y = 0 , \tag{6.1}$$

where primes now denote differentiation with respect to t. The initial conditions are

$$Y(0) = 0, \quad Y'(0) = 1 , \tag{6.2}$$

and we consider in tandem with (6.1) the equation

$$Y_0'' + \frac{2}{t} Y_0' + \{Y_0 - \frac{2}{t^2}\} Y_0 = 0 , \tag{6.3}$$

with the same initial conditions

$$Y_0(0) = 0, \quad Y_0'(0) = 1 .$$

If we can show that $Y_0(t)$ becomes negative for some finite t, then since (for sufficiently large α) the solution of (6.1)-(6.2) can be made as close as we please to $Y_0(t)$ in any compact t-interval, it will follow that $Y(t)$ also becomes negative, and the lemma is proved.

To prove that Y_0 becomes negative, we note that Y_0, Y'_0 are initially positive, and that $t^2Y'_0$ is increasing until Y_0 meets the function $2/t^2$. Thus Y_0 must meet $2/t^2$ (at a finite point), since $2/t^2$ is decreasing.

We now ask whether Y_0, after passing above $2/t^2$, meets $2/t^2$ again. Suppose for contradiction that it does not. Then $t^2Y'_0$ is decreasing, and it must become negative, since otherwise Y_0 is increasing and (6.3) implies that Y''_0 becomes strictly negative, which ultimately forces Y'_0 to be negative. We thus have Y_0 positive and Y'_0 negative, and so the solution is bounded and exists for all t, and $t^2Y'_0 \to L$ (say) as $t \to \infty$, with $L < 0$. We shall suppose L finite, a similar argument applying if $L = -\infty$. Thus $Y'_0 \sim L/t^2$, $Y_0 \to M$ (say), M finite, and it is clear from (6.3) that the only possible value for M is $M = 0$. Thus

$$Y_0 \sim -L/t \quad ,$$

and (6.3) gives

$$(t^2Y'_0)' \sim -L^2 \quad ,$$

$$Y'_0 \sim -L^2/t \quad ,$$

which is a contradiction.

Thus Y_0 meets $2/t^2$ a second time, and at the point of meet we must have

$$Y_0 = 2/t^2, \qquad Y'_0 \leq -4/t^3 \quad .$$

Now the solution of the equation

$$z'' + \frac{2}{t} z' - \frac{2}{t^2} z = 0$$

with the initial conditions (at some point t_0)

$$z(t_0) = 2/t_0^2, \quad z'(t_0) = -4/t_0^3$$

is precisely the function $2/t^2$, which remains positive but tends to zero as $t \to \infty$, and comparing the equations for Y_0 and $z = 2/t^2$, we have

$$(Y_0 - z)'' + \frac{2}{t}(Y_0 - z)' - \frac{2}{t^2}(Y_0 - z) = -Y_0^2 ,$$

whence, as in Section 2,

$$Y_0 - z = At + \frac{B}{t^2} - \frac{1}{3} \int_{t_0}^{t} (t - \frac{u^3}{t^2}) Y_0^2(u) du$$

with the conditions

$$(Y_0 - z)(t_0) = 0, \quad (Y_0 - z)'(t_0) \leq 0 .$$

This leads easily to the inequalities $A \leq 0$, $B \geq 0$, and so, as $t \to \infty$,

$$Y_0 - z = At + \frac{B}{t^2} - L(t), \quad \text{say,}$$

where $L(t)$ is positive increasing. Since $A \leq 0$, this certainly implies that Y_0 becomes negative and completes the proof of the lemma.

7. Proof of Lemma 6

Arguing as in Lemma 4, we see that the lemma will be proved if we can show that, for α sufficiently small, there is some r_0, possibly depending on α, for which

$$G(r_0, \alpha) < 0 \quad . \tag{7.1}$$

In order to establish the existence of such an r_0, we write (1.1) in the form

$$y'' + \frac{2}{r} y' - (1 + \frac{2}{r^2}) y = -y^2 \quad , \tag{7.2}$$

and note that the solutions of

$$y'' + \frac{2}{r} y' - (1 + \frac{2}{r^2}) y = 0$$

are

$$f(r) = \frac{e^r}{r}(1 - \frac{1}{r}), \quad g(r) = \frac{e^{-r}}{r}(1 + \frac{1}{r}) \quad .$$

Using the variation of constants formula, we can write (7.2) in the form (taking account of the initial conditions)

$$y = \frac{3}{2}\alpha(f+g) - \frac{1}{2}\int_0^r \{f(r)\,g(t) - f(t)\,g(r)\} t^2 y^2(t) dt \quad ,$$

from which the usual iteration process assures us that

$$y - \frac{3}{2}\alpha(f + g) = O(\alpha^2) \quad \text{as} \quad \alpha \to 0 \quad ,$$

the estimate being uniform in any fixed interval $[0, R]$. We shall therefore certainly have established (7.1) if we can show that there exists a fixed r_0

(independent of α) for which (7.1) holds for α sufficiently small in the form

$$G(r_0, \alpha) < -K\alpha^2 ,$$

where K is a positive constant independent of α and, in evaluating G, we set $y = \frac{3}{2}\alpha(f+g)$. Since the term $\frac{1}{3}y^3$ in G is $O(\alpha^3)$, it will be sufficient to prove that there exists a fixed r_0 for which

$$\frac{1}{2}y'^2(r_0) - \frac{1}{2}y^2(r_0) - \frac{y^2(r_0)}{r_0^2} + \frac{2}{r_0}\int_0^{r_0} y'^2(t)dt < -K\alpha^2 \qquad (7.3)$$

for α sufficiently small, with $y = \frac{3}{2}\alpha(f+g)$. In evaluating the left-hand side of (7.3), we can neglect the factor $(\frac{3}{2}\alpha)^2$ which is common to every term, and if we replace $f + g$ by just $f = \frac{e^r}{r}(1 - \frac{1}{r})$, then we neglect only terms which (for large r_0) are at worst polynomial in $1/r_0$. With therefore $y = \frac{e^r}{r}(1 - \frac{1}{r})$, it is standard to show that, as $r \to \infty$,

$$\frac{1}{2}y'^2 - \frac{1}{2}y^2 - \frac{y^2}{r^2} + \frac{2}{r}\int_0^r y'^2(t)dt \sim -\frac{3}{2}\frac{e^{2r}}{r^4} ,$$

which certainly implies that, if r_0 is fixed sufficiently large, then the term arising from $-\frac{3}{2}e^{2r_0}/r_0^4$ on the left of (7.3) will dominate any of the other terms, and so prove (7.3) and the lemma.

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The existence of a non-negative solution of the boundary-value problem

$$y'' + \frac{2}{r}y' + [y - (1 + \frac{2}{r^2})]y = 0, \qquad y(0) = 0, \qquad y(\infty) = 0 ,$$

is proved by a shooting argument. The equation arises in an electromagnetic theory for strong interaction in charged media.

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